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*Publication date:*  
1996

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*Citation for published version (APA):*

van Dam, E. R., & Spence, E. (1996). *Small Regular Graphs with Four Eigenvalues*. (FEW Research Memorandum; Vol. 723). Operations research.

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# Small regular graphs with four eigenvalues

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**Abstract.** For most feasible spectra of connected regular graphs with four distinct eigenvalues and at most 30 vertices we find all such graphs, using both theoretic and computer results.

## 1. Introduction

Graphs with a few eigenvalues in general have nice combinatorial properties and a rich structure. A well-investigated family of such graphs comprises the strongly regular graphs (the regular graphs with three eigenvalues) and these too have a nice combinatorial characterization. Two combinatorial generalizations of these strongly regular graphs are the distance-regular graphs, and going a step further, the association schemes (cf. [1]). The first stage of investigation after strongly regular graphs would be to consider three-class association schemes (cf. [5]). In such a scheme all graphs are regular with at most four eigenvalues, but of course not all graphs with four eigenvalues are in a three-class association scheme (indeed most are not). Even so, as mentioned earlier, such graphs have some interesting combinatorial properties and were studied previously by the first author [4]. Using the results from [4] we generated a list of feasible spectra for regular graphs with four eigenvalues and at most 30 vertices (for the definition of feasible, see Section 2). Using both theoretic and computer results we were able to find *all* graphs when a graph did exist, or show that none exists, for 214 of the 244 feasible spectra thus found. To be precise, we know that there does not exist a graph for 68 of these feasible spectra. In 10 of the remaining 30 cases whose classification is not yet finished, we have found some graphs after an incomplete computer search, while the other 20 cases are completely open. Of these, the smallest unsolved case is one on 24 vertices.

Sections 2, 3 and 4 of this paper contain results from [4] that are relevant to our search, while Section 5 is used to prove the non-existence of graphs for a considerable number of the feasible spectra. Section 6 contains an explanation of the methods used in the computer search and the Appendix includes the lists of feasible spectra together with the number of graphs for each spectrum. Moreover, all new *transitive* graphs found by computer are also there. In the case of a spectrum for which new graphs were discovered,

none of which was transitive, we also include as a representative one with the largest automorphism group.

## 2. Feasible spectra

If  $G$  is a connected regular graph on  $v$  vertices with four distinct eigenvalues, then (cf. [4])

- i.  $G$  has four integral eigenvalues, or
- ii.  $G$  has two integral eigenvalues, and two eigenvalues of the form  $\frac{1}{2}(a \pm \sqrt{b})$ , with  $a, b \in \mathbb{Z}$ ,  $b > 0$ , with the same multiplicity, or
- iii.  $G$  has one integral eigenvalue, its degree  $k$ , and the other three have the same multiplicity  $m = \frac{1}{3}(v - 1)$ , and  $k = m$  or  $k = 2m$ .

In addition, if  $G$  has  $v$  vertices and spectrum  $\{[\lambda_0 = k]^1, [\lambda_1]^{m_1}, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}\}$ , then the following three equations uniquely determine the multiplicities  $m_1, m_2$  and  $m_3$  from  $v$  and the eigenvalues  $\lambda_0, \lambda_1, \lambda_2, \lambda_3$  (cf. [3, 4]).

$$\begin{aligned} 1 + m_1 + m_2 + m_3 &= v, \\ \lambda_0 + m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 &= 0, \\ \lambda_0^2 + m_1\lambda_1^2 + m_2\lambda_2^2 + m_3\lambda_3^2 &= vk. \end{aligned}$$

The second equation follows from the trace of  $A$  (the adjacency matrix of  $G$ ), and the third from the trace of  $A^2$ . Using these conditions we were able to generate all possible spectra for regular graphs with four eigenvalues and at most 30 vertices. Different algorithms were used in each of the three cases above and they in turn were checked for the following further conditions.

Since a regular graph with four eigenvalues is walk-regular (cf. [4]), it follows that the number of triangles through a given vertex  $x$  is independent of  $x$ , and equals

$$\Delta = \frac{1}{2}A_{xx}^3 = \frac{\text{Trace}(A^3)}{2v} = \frac{1}{2v} \sum_{i=0}^3 m_i \lambda_i^3.$$

This expression gives a further feasibility condition for the spectrum of  $G$  since  $\Delta$  should be a non-negative integer. In general, it follows that

$$\theta_r = \frac{1}{v} \sum_{i=0}^3 m_i \lambda_i^r$$

is a non-negative integer. Since the number of closed walks of odd length  $r$  is even,  $\theta_r$  should be even if  $r$  is odd. For even  $r$  we can sharpen the condition since the number of

non-trivial closed walks (those containing a cycle) is even. When  $r = 4$  the number of trivial closed walks through a given vertex (i.e. passing through one or two other vertices only) equals  $2k^2 - k$ , and this means that

$$\Xi = \frac{\theta_4 - 2k^2 + k}{2}$$

is a non-negative integer, and it equals the number of quadrangles through a vertex. When  $r = 6$ , the number of non-trivial closed walks through a vertex equals  $\theta_6 - k(5k^2 - 6k + 2)$ , which should be even.

The complement of a connected regular graph with four eigenvalues is also such a graph unless it is disconnected. By generating only those spectra for which  $k \geq v - 1 - k$ , we ensured that any putative graph would be connected, but in the Appendix we printed the complementary spectrum, unless it implied disconnectivity.

In the algorithm to generate spectra with four integral eigenvalues we checked that  $\theta_r$  was an even non-negative integer for  $r = 3, 5, 7, 9$  and  $11$  and that it was a non-negative integer in the cases  $r = 8, 10$  and  $12$ . In addition we tested to see that both  $\theta_4 - 2k^2 + k$  and  $\theta_6 - k(5k^2 - 6k + 2)$  were even non-negative integers, and that the complementary spectrum gave rise to numbers of triangles and quadrangles through a vertex that were also non-negative integers. For technical reasons we checked different conditions in the case of two integral eigenvalues, namely the conditions on  $\theta_r$  for  $r = 3, \dots, 6$  and the complementary  $\theta_r$  for  $r = 3, \dots, 8$ . Finally, in the remaining case of one integral eigenvalue it was not necessary to implement so many conditions. Here we checked only the conditions on  $\theta_3$  and  $\theta_4$ . When a putative spectrum satisfies all of the above conditions, it is termed *feasible*.

### 3. Special spectral properties

#### 3.1. A useful idea

Let  $G$  be a connected  $k$ -regular graph on  $v$  vertices with four eigenvalues  $k, \lambda_1, \lambda_2$  and  $\lambda_3$ . If  $G$  has a simple eigenvalue, besides  $k$ , say  $\lambda_3$ , then  $G$  admits a regular partition into halves with degrees  $(\frac{1}{2}(k + \lambda_3), \frac{1}{2}(k - \lambda_3))$ , that is, we can partition the vertices into two parts of equal size such that every vertex has  $\frac{1}{2}(k + \lambda_3)$  neighbours in its own part and  $\frac{1}{2}(k - \lambda_3)$  neighbours in the other part (cf. [4]). A consequence of this is that  $k - \lambda_3$  is even, a condition which was proved by Godsil and McKay [6]. This condition eliminates the existence of a graph with spectrum  $\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$ . Moreover,  $v$  is a divisor of

$$(k - \lambda_1)(k - \lambda_2) + (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \text{ and } (k - \lambda_1)(k - \lambda_2) - (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2),$$

from which we derive that there is no graph with spectrum  $\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$ .

It is possible to prove all this, using an idea on which most of our results, both theoretic and computer, are built. Let  $A$  be the adjacency matrix of  $G$ . The matrix  $C = C(\lambda_1, \lambda_2)$  defined by

$$C(\lambda_1, \lambda_2) = (A - \lambda_1 I)(A - \lambda_2 I) - \frac{(k - \lambda_1)(k - \lambda_2)}{v} J$$

is a symmetric matrix with row sums zero and one non-zero eigenvalue  $(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$  with multiplicity  $m_3$  (the multiplicity of  $\lambda_3$  as an eigenvalue of  $G$ ). Now  $C$  or  $-C$  is a positive semidefinite matrix of rank  $m_3$ , and has constant diagonal  $k + \lambda_1\lambda_2 - (k - \lambda_1)(k - \lambda_2)/v$ . Of course, as  $A^2$  is a matrix with non-negative integral entries and  $A$  is a  $(0, 1)$ -matrix, the other entries of  $C$  are very restricted. Especially when  $m_3$  is small we get strong restrictions on the structure of  $G$ . This enables us to show uniqueness of the graph in Proposition 1, and prove the non-existence of graphs in a substantial number of cases in Section 5. It also proved to be a powerful tool in our computer search, as is explained in more detail in Section 6.

### 3.2. Disconnected complement

If  $G$  has  $v$  vertices, degree  $k$ , and an eigenvalue  $k - v$ , then the complement of  $G$  is disconnected, and then it must be the disjoint union of strongly regular graphs with the same spectrum and hence with the same parameters.

### 3.3. Graphs with least eigenvalue at least $-2$

If  $G$  has least eigenvalue at least  $-2$ , then it must be  $C_7$ , or the line graph of some graph  $H$ , where  $H$  is a strongly regular graph, or  $H$  is the incidence graph of a symmetric design, or  $H$  is a complete bipartite graph, or  $G$  is one of the graphs found by Bussemaker, Cvetković and Seidel [2]:

$\text{BCS}_9$	: one graph on 12 vertices with spectrum	$\{[4]^1, [2]^3, [0]^3, [-2]^5\}$ ,
$\text{BCS}_{70}$	: one graph on 18 vertices with spectrum	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$ ,
$\text{BCS}_{153}$ - $\text{BCS}_{160}$	: eight graphs on 24 vertices with spectrum	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$ ,
$\text{BCS}_{179}$	: one graph on 18 vertices with spectrum	$\{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$ ,
$\text{BCS}_{183}$	: one graph on 24 vertices with spectrum	$\{[14]^1, [4]^4, [2]^2, [-2]^{17}\}$ .

Note that the complement of a connected regular graph with least eigenvalue  $-2$ , is a graph with second largest eigenvalue 1.

## 4. Constructions and small examples

### 4.1. Distance-regular graphs and association schemes

In [7] (almost) all graphs with the spectrum of a distance-regular graph with at most 30 vertices are found. Most of these graphs have four distinct eigenvalues.

A substantial number of examples of distance-regular graphs with four eigenvalues are given by the incidence graphs of symmetric designs. In fact, any bipartite graph with four eigenvalues must be the incidence graph of a symmetric design, and hence is distance-regular (cf. [3, p. 166]).

Some other graphs can be obtained by merging classes in distance-regular graphs (cf. [1]) or association schemes. By  $G_i$  we denote the distance  $i$  graph of  $G$ . For example, the distance 3 together with the distance 5 relation in the Dodecahedron gives a graph with four eigenvalues. Also the distance 4 relation in the Coxeter graph gives such a graph.

In the tables of the Appendix we have added, for every spectrum, the number of graphs with that spectrum or complementary spectrum that are a relation in some three-class association scheme. These numbers are obtained from [5].

#### 4.1.1. Pseudocyclic association schemes

A three-class association scheme is said to be pseudocyclic if there are three eigenvalues with the same multiplicity. If the number of vertices  $q$  is a prime power and  $q \equiv 1 \pmod{3}$ , then the cyclotomic scheme, which has the third power cyclotomic classes of  $GF(q)$  as classes, is an example. For  $q > 4$ , the corresponding graph  $\text{Cycl}(q)$  has four distinct eigenvalues and is obtained by making two elements of  $GF(q)$  adjacent if their difference is a cube. The smallest example is the 7-cycle  $C_7$ . It is determined by its spectrum, as are  $\text{Cycl}(13)$  and  $\text{Cycl}(19)$ , which we can prove by hand. On 28 vertices two schemes are known. Mathon [9] found one, and Hollmann [8] proved that there are precisely two.

### 4.2. Product constructions

If  $G$  is a graph with adjacency matrix  $A$ , then we denote by  $G \otimes J_n$  the graph with adjacency matrix  $A \otimes J_n$  ( $\otimes$  denotes the Kronecker product), and by  $G \oplus J_n$  the graph with adjacency matrix  $(A + I) \otimes J_n - I$ . Note that  $\overline{G \otimes J_n} = \overline{G} \oplus J_n$ , where  $\overline{G}$  is the complement of  $G$ . The first construction adds an eigenvalue 0 to the spectrum, while the second construction adds an eigenvalue  $-1$ .

Thus, if we have a strongly regular graph or a connected regular graph with four distinct eigenvalues of which one is 0 or  $-1$ , then this construction produces a bigger graph with four distinct eigenvalues.

For any  $n$ ,  $C_5 \otimes J_n$  and  $C_5 \oplus J_n$  are uniquely determined by their spectra. Furthermore, if

$IG(l, l-1, l-2)$  denotes the incidence graph of the unique (trivial)  $2$ -( $l, l-1, l-2$ ) design, then for each  $l$  and  $n$ , the graph  $IG(l, l-1, l-2) \oplus J_n$  is uniquely determined by its spectrum (cf. [4]).

If  $A$  is the adjacency matrix of a graph  $G$ , then the graph with adjacency matrix

$$\begin{pmatrix} A & I \\ I & J - I - A \end{pmatrix}$$

is called the twisted double  $\text{td}G$  of  $G$ . Now let  $v = 4\mu + 1$  and  $k = 2\mu$ . Then  $G$  is a graph with spectrum  $\{[k+1]^1, [k-1]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{v+4}]^{2k}, [-\frac{1}{2} - \frac{1}{2}\sqrt{v+4}]^{2k}\}$  if and only if  $G$  is the twisted double of a conference graph on  $v$  vertices (cf. [4]). Since the conference graphs  $L_2(3)$  on 9 vertices and  $P(13)$  on 13 vertices are unique, also their twisted doubles are uniquely determined by their spectra.

Let  $G$  and  $G'$  be graphs with adjacency matrices  $A$  and  $A'$ , and eigenvalues  $\lambda_i$ ,  $i = 1, 2, \dots, v$ , and  $\lambda'_j$ ,  $j = 1, 2, \dots, v'$ , respectively. Then the graph with adjacency matrix  $A \otimes I_{v'} + I_v \otimes A'$  has eigenvalues  $\lambda_i + \lambda'_j$ ,  $i = 1, 2, \dots, v$ ,  $j = 1, 2, \dots, v'$ . This graph, which is sometimes called the sum [3] or the Cartesian product of  $G$  and  $G'$ , will be denoted by  $G \oplus G'$ . An example with four distinct eigenvalues is  $G \oplus K_m$ , where  $G$  is the complete bipartite graph  $K_{m,m}$  or the lattice graph  $L_2(m)$ . Here we present our first new result.

**PROPOSITION 1.** *The graph  $K_{3,3} \oplus K_3$  is uniquely determined by its spectrum.*

*Proof.* Let  $G$  be a graph with spectrum  $\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$  and adjacency matrix  $A$ . Then  $G$  is a 5-regular graph on 18 vertices with one triangle through each vertex. The matrix  $C = C(2, -1) = A^2 - A - 2I - J$ , as defined in Section 3.1 is a positive semidefinite integral matrix of rank two with diagonal 2. Thus  $C$  is the Gram matrix of a set of vectors in  $\mathbb{R}^2$  of length  $\sqrt{2}$  such that their inner products are  $\pm 2, \pm 1$  or 0.

Note that if two vertices are adjacent and the vectors representing these vertices have inner product  $-1$ , then they are in a triangle. This implies that any vertex is adjacent to precisely two vertices such that their inner product is  $-1$ , and that the inner product between those two vertices is also  $-1$ . If two vertices have inner product  $-2$  then they are adjacent, and if they have inner product 1 or 2 then they are not adjacent.

Without loss of generality we assume that there is a vertex represented by vector  $\sqrt{2}(1, 0)^T$ . This vertex must be in a triangle with vertices represented by vectors  $\sqrt{2}(-\frac{1}{2}, \frac{1}{2}\sqrt{3})^T$  and  $\sqrt{2}(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})^T$ . Furthermore it is adjacent to three vertices represented by  $\sqrt{2}(-1, 0)^T$ . In turn, such a vertex is in a triangle with vertices represented by vectors  $\sqrt{2}(\frac{1}{2}, \frac{1}{2}\sqrt{3})^T$  and  $\sqrt{2}(\frac{1}{2}, -\frac{1}{2}\sqrt{3})^T$ , and is adjacent to three vertices represented by  $\sqrt{2}(1, 0)^T$ .

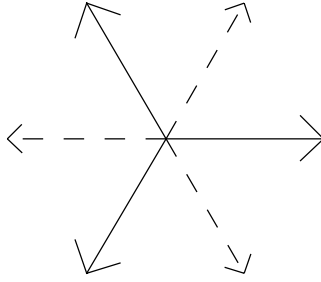


Figure 1. Vectors representing the vertices of  $K_{3,3} \oplus K_3$

In this way we find 18 vertices: each of the 6 mentioned vectors represents 3 vertices. Now, up to isomorphism, all adjacencies follow from the inner products and the fact that every vertex is in one triangle. The graph we obtain is  $K_{3,3} \oplus K_3$ .  $\square$

#### 4.3. Examples from strongly regular graphs

If  $G$  is a non-bipartite strongly regular graph on  $v$  vertices, with spectrum  $\{[k]^1, [r]^f, [s]^g\}$ , and  $C$  is a coclique of size  $c$  meeting the Hoffman bound, i.e.  $c = -vs/(k - s)$ , then the induced subgraph  $G \setminus C$  on the vertices not in  $C$  is a regular, connected graph with four eigenvalues if  $c < g$ . By looking at the complement of the graph, a similar construction works for cliques instead of cocliques. An example is obtained by removing a 3-clique (a line) in the generalized quadrangle  $\text{GQ}(2, 2)$ , and the resulting graph has spectrum  $\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$ .

If  $G$  contains a spread, that is, a partition of the vertices into Hoffman cliques, then by removing the edges of this spread, we also obtain a regular graph with four eigenvalues. For example, the complement of the triangular graph  $T(n)$  contains (many) spreads for every even  $n$ .

Also some subconstituents (i.e. induced subgraphs on the set of (non-)neighbours of a given vertex) of strongly regular graphs have four distinct eigenvalues, for example  $\text{GQ}(3, 3)_2(x)$  (cf. [4]).

#### 4.4. Covers

In [4]  $n$ -covers of  $C_3 \otimes J_n$ ,  $C_3 \circledast J_n = K_{3n}$ ,  $C_5 \otimes J_n$ ,  $C_6 \otimes J_n$  and  $\text{Cube} \otimes J_n$ , having four distinct eigenvalues are constructed. The 2-cover of  $C_3 \otimes J_2$  is isomorphic to the line graph of the Cube, the 2-cover of  $C_3 \otimes J_2$  is isomorphic to the Icosahedron, and the 3-cover of  $C_3 \otimes J_3$  is cospectral but not isomorphic to  $H(3, 3)$ . The graphs in Figure 2 are the three remaining covers with at most 30 vertices. They were shown, using the computer, to be uniquely determined by their spectra.



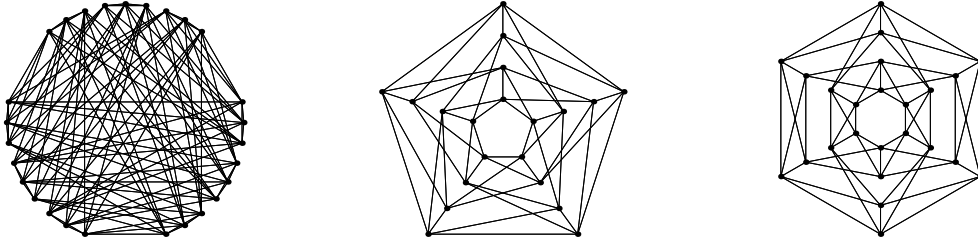


Figure 2. A 3-cover of  $C_3 \otimes J_3$  and 2-covers of  $C_5 \otimes J_2$  and  $C_6 \otimes J_2$

## 5. Non-existence results

In this section we examine several feasible spectra and prove the non-existence of a graph in each case. In each of the following propositions we assume the existence of a graph  $G$  with the given spectrum and  $A$  will denote its adjacency matrix.

Let  $G$  be a  $k$ -regular graph on  $v$  vertices with  $\Delta$  triangles and  $\Xi$  quadrangles through every vertex. Fix a vertex  $x$ , and let  $\sigma_i$  be the number of vertices  $y$  adjacent to  $x$ , such that  $A^2_{xy} = i$ , and let  $\tau_i$  be the number of vertices  $y$  not adjacent to  $x$ , such that  $A^2_{xy} = i$ . Then counting arguments show that

$$\sum_i \sigma_i = k, \sum_i i \sigma_i = 2\Delta, \sum_i \tau_i = v - k - 1, \sum_i i \tau_i = k(k-1) - 2\Delta, \text{ and } \sum_i \binom{i}{2} (\sigma_i + \tau_i) = \Xi.$$

We shall call this system of equations the  $(\sigma, \tau)$ -system.

**PROPOSITION 2.** *There is no graph with spectrum  $\{[7]^1, [2]^{15}, [-2]^5, [-3]^9\}$ .*

*Proof.* Such a graph would be 7-regular on 30 vertices with  $\Delta = 3$  triangles and  $\Xi = 12$  quadrangles through every vertex. Let  $C = C(2, -3) = A^2 + A - 6I - \frac{5}{3}J$ , then  $-C$  is a positive semidefinite matrix with diagonal  $\frac{2}{3}$ . It follows that  $C$  can only have entries  $-\frac{2}{3}$  and  $\frac{1}{3}$ , and so if  $x$  and  $y$  are adjacent then  $A^2_{xy} = 0$  or 1, and if  $x$  and  $y$  are not adjacent then  $A^2_{xy} = 1$  or 2. But now the  $(\sigma, \tau)$ -system does not have a solution, so we have a contradiction.  $\square$

Similarly we can prove that there are no graphs with spectra  $\{[6]^1, [2]^9, [1]^9, [-3]^{11}\}$ ,  $\{[7]^1, [2]^{12}, [1]^5, [-3]^{12}\}$ ,  $\{[6]^1, [1 + \sqrt{10}]^2, [-1]^{10}, [1 - \sqrt{10}]^2\}$ ,  $\{[7]^1, [1 + 2\sqrt{3}]^2, [-1]^{11}, [1 - 2\sqrt{3}]^2\}$  and  $\{[8]^1, [-1 + \sqrt{6}]^7, [1]^6, [-1 - \sqrt{6}]^7\}$ .

PROPOSITION 3. *There is no graph with spectrum  $\{[8]^1, [2 + 3\sqrt{2}]^3, [-1]^{20}, [2 - 3\sqrt{2}]^3\}$ .*

*Proof.* Here we have a 8-regular graph on 27 vertices with  $\Delta = 22$  triangles and  $\Xi = 102$  quadrangles through every vertex. The matrix  $C$  as defined in Section 3.1 by  $C = C(2 + 3\sqrt{2}, -1) = A^2 - (1 + 3\sqrt{2})A - (2 + 3\sqrt{2})I - (2 - \sqrt{2})J$ , is a positive semidefinite matrix with diagonal  $4 - 2\sqrt{2}$ . It follows that if  $x$  and  $y$  are adjacent then  $A^2_{xy} = 5, 6$  or  $7$ , and if  $x$  and  $y$  are not adjacent then  $A^2_{xy} = 0$  or  $1$ . Now the  $(\sigma, \tau)$ -system has one solution  $\sigma_7 = 2, \sigma_6 = 0, \sigma_5 = 6, \tau_1 = 12, \tau_0 = 6$ . But then  $G = H \otimes J_3$ , for some graph  $H$ . It follows that  $H$  must have spectrum  $\{[2]^1, [\sqrt{2}]^3, [-1]^2, [-\sqrt{2}]^3\}$ , but since such a graph does not exist, we have a contradiction.  $\square$

We can also prove that a graph with spectrum  $\{[9]^1, [7]^3, [-1]^{24}, [-3]^2\}$  does not exist by showing that such a graph would be of the form  $H \otimes J_2$ , where  $H$  has spectrum  $\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$ .

The next proposition uses the fact that if the number of triangles through an edge is constant, then the number of quadrangles through an edge is also constant (cf. [4]). If  $\xi$  is the (constant) number of quadrangles through an edge, and if  $\Xi$  is the number of quadrangles through a vertex, then  $\xi = 2\Xi/k$ .

PROPOSITION 4. *There are no graphs with spectra  $\{[8]^1, [-1 + \sqrt{21}]^4, [0]^{21}, [-1 - \sqrt{21}]^4\}$  or  $\{[4]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{21}]^4, [0]^6, [-\frac{1}{2} - \frac{1}{2}\sqrt{21}]^4\}$ .*

*Proof.* Note that if  $H$  is a graph with the second spectrum, then  $H \otimes J_2$  is a graph with the first spectrum. Thus it suffices to show that there is no graph with the first spectrum.

Suppose  $G$  is such a graph, then  $G$  is 8-regular on 30 vertices without triangles, such that every vertex is in  $\Xi = 84$  quadrangles and every edge is in  $\xi = 21$  quadrangles.

Suppose first of all that  $G$  has diameter 2. Take two non-adjacent vertices  $x$  and  $z$  such that  $A^2_{xz} = 1$  and let  $y$  be their common neighbour. Now the 21 quadrangles through  $\{x, y\}$  and the 21 quadrangles through  $\{z, y\}$  are distinct, and since there are 42 edges between  $G(y) \setminus \{x, z\}$  and  $G_2(y)$ , all these edges contain a neighbour of  $x$  or  $z$ . Then it follows that the number of vertices at distance 2 from  $y$  is 14, and so  $G$  has diameter 3, which is a contradiction. Thus for any two non-adjacent vertices  $x$  and  $z$  we must have  $A^2_{xz} \geq 2$ . But then the  $(\sigma, \tau)$ -system has no non-negative integral solution. Thus  $G$  has diameter 3.

Take a vertex  $x$  and let  $y$  be a vertex at distance 3 from  $x$ . Let  $A$  be partitioned into two parts, where one part contains  $y$  and the neighbours of  $x$ . Then

$$A = \begin{pmatrix} O_{9,9} & N \\ N^T & B \end{pmatrix}.$$

Since  $\text{rank}(A) = 9$ , it follows that  $\text{rank}(N) \leq 4$ . Now write

$$N = \begin{pmatrix} 1 & N_1 \\ -8 & N_2 \end{pmatrix}, \text{ and } N' = \begin{pmatrix} 0 & N_1 \\ 1 & N_2 \end{pmatrix}.$$

Since the all-one vector is in the column space of  $N$  ( $N$  has constant row sums 8),  $\text{rank}(N') \leq \text{rank}(N)$ , so  $\text{rank}(N_1) \leq 3$ . Moreover,  $N_1$  has constant row sums 7, and so it follows that  $N_1$  is of the form

$$N_1 = \begin{pmatrix} J_{m_1, 7-t_1-t_2} & J_{m_1, t_1} & O_{m_1, t_1} & J_{m_1, t_2} & O_{m_1, t_2} & O_{m_1, 13-t_1-t_2} \\ J_{m_2, 7-t_1-t_2} & J_{m_2, t_1} & O_{m_2, t_1} & O_{m_2, t_2} & J_{m_2, t_2} & O_{m_2, 13-t_1-t_2} \\ J_{m_3, 7-t_1-t_2} & O_{m_3, t_1} & J_{m_3, t_1} & J_{m_3, t_2} & O_{m_3, t_2} & O_{m_3, 13-t_1-t_2} \\ J_{m_4, 7-t_1-t_2} & O_{m_4, t_1} & J_{m_4, t_1} & O_{m_4, t_2} & J_{m_4, t_2} & O_{m_4, 13-t_1-t_2} \end{pmatrix},$$

with  $m_1 + m_2 + m_3 + m_4 = 8$ , and  $t_1, t_2 \neq 0$ , or that  $N_1$  has at most 3 distinct rows.

Suppose we are in the first case. If we count the number of quadrangles through  $x$  and a vertex  $z$  which corresponds to one of the first  $m_1$  rows, then we see that

$$\xi = 7(m_1 - 1) + (7 - t_2)m_2 + (7 - t_1)m_3 + (7 - t_1 - t_2)m_4.$$

If we count the number of quadrangles through  $x$  and a vertex corresponding to one of the  $m_2$  rows of the second block, then

$$\xi = 7(m_2 - 1) + (7 - t_2)m_1 + (7 - t_1)m_4 + (7 - t_1 - t_2)m_3.$$

From this it follows that  $m_1 + m_3 = m_2 + m_4 = 4$  and  $t_1 + t_2 = 7$ . Similarly it follows that  $m_1 + m_2 = m_3 + m_4 = 4$ , and so that  $m_1 = m_4$  and  $m_2 = m_3$ . This implies that  $G_3(x)$  has 7 vertices and that every vertex in  $G_2(x)$  has 4 neighbours in  $G(x)$ . From the Hoffman polynomial it follows that if  $y$  is a vertex at distance 3 from  $x$ , then  $A_{xy}^3 = 16$ , so in turn every vertex in  $G_3(x)$  has 4 neighbours in  $G_2(x)$ . But then the induced subgraph on  $G_3(x)$  is 4-regular on 7 vertices, and this is not possible without triangles.

Thus we are in the second case. Suppose  $N_1$  has 4 identical rows. By counting the number of quadrangles through  $x$  and a vertex corresponding to one of these 4 rows it follows that the other 4 rows are disjoint from the first 4. Further counting gives that the other 4 rows must also be the same, and again we have that  $G_3(x)$  has 7 vertices and that every vertex in  $G_2(x)$  has 4 neighbours in  $G(x)$ , which leads to a contradiction.

It follows that we have one row occuring twice and two rows occuring three times. By counting quadrangles through  $x$  and a vertex corresponding to one of the rows occuring twice, we see that

$$\xi = 7 + 3t_1 + 3t_2,$$

for some  $t_1, t_2$ , and so 14 should be divisible by 3, which is a contradiction.  $\square$

Next we shall prove the non-existence of some graphs, assuming that they have an eigenvalue with multiplicity 2.

PROPOSITION 5. *There is no graph with spectrum  $\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$ .*

*Proof.* Such a graph is 7-regular on 24 vertices with 5 triangles through each vertex. Let  $C = C(3, -1) = A^2 - 2A - 3I - \frac{4}{3}J$ , then  $C$  is a positive semidefinite matrix of rank two with row sums zero and diagonal  $\frac{8}{3}$ . Thus  $C$  can only have entries  $-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{5}{3}$  and  $\frac{8}{3}$ .

Now suppose that  $C_{xy} = -\frac{1}{3}$  for some vertices  $x$  and  $y$ . Let  $z$  be another arbitrary vertex. Since  $C$  has rank two it follows that the principal submatrix of  $C$  on vertices  $x, y$  and  $z$  has zero determinant, and so either  $C_{xz} = \frac{8}{3}$  and  $C_{yz} = -\frac{1}{3}$  or  $C_{xz} = -\frac{1}{3}$  and  $C_{yz} = \frac{8}{3}$ . But then  $x$  and  $y$  cannot both have row sums zero, and it follows that  $C$  has no entries  $-\frac{1}{3}$ . Similarly it follows that  $C$  cannot have entries  $\frac{5}{3}, -\frac{7}{3}$  and  $\frac{2}{3}$ . Thus  $C$  can only have entries  $\frac{8}{3}$  and  $-\frac{4}{3}$ .

Now fix  $x$ . For all vertices  $y$  adjacent to  $x$ , we must have  $A^2_{xy} = 2$  or  $6$ . But  $x$  has 7 neighbours, giving that  $x$  is in at least 7 triangles, which is a contradiction.  $\square$

Similarly it can be proved that there are no graphs with spectrum  $\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$ ,  $\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$  and  $\{[9]^1, [4]^6, [-1]^{21}, [-6]^2\}$ .

PROPOSITION 6. *There is no graph with spectrum  $\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$ .*

*Proof.* Here we would have a 12-regular graph on 27 vertices with  $\Delta = 6$  triangles and  $\Xi = 492$  quadrangles through every vertex. The matrix  $C(0, -9)$  is positive semidefinite of rank two with row sums zero and diagonal  $\frac{8}{3}$ . Thus  $C$  can only have entries  $-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{5}{3}$  and  $\frac{8}{3}$ .

Now suppose that  $C_{xy} = -\frac{1}{3}$  for some vertices  $x$  and  $y$ . Let  $z$  be another arbitrary vertex. Since  $C$  has rank two it follows that the principal submatrix of  $C$  on vertices  $x, y$  and  $z$  has zero determinant, and so either  $C_{xz} = \frac{8}{3}$  and  $C_{yz} = -\frac{1}{3}$  or  $C_{xz} = -\frac{1}{3}$  and  $C_{yz} = \frac{8}{3}$ . But then  $x$  and  $y$  cannot both have row sums zero. Thus  $C$  has no entries  $-\frac{1}{3}$ . This implies that if  $x$  and  $y$  are adjacent then  $A^2_{xy} \neq 0$ , and since there are only 6 triangles through every vertex, it follows that  $A^2_{xy} = 1$  ( $\sigma_1 = 12$ ), and so  $C_{xy} = \frac{2}{3}$ . Again, let  $z$  be another vertex, then it follows that  $C_{xz} = \frac{2}{3}, \frac{8}{3}$  or  $-\frac{7}{3}$ . Now it follows that if  $x$  and  $z$  are not adjacent, then  $A^2_{xz} = 7, 10$  or  $12$ . But then the  $(\sigma, \tau)$ -system has no integral solution, giving a contradiction.  $\square$

PROPOSITION 7. *There is no graph with spectrum  $\{[9]^1, [3]^8, [-1]^{19}, [-7]^2\}$ .*

*Proof.* Now our graph would be 9-regular on 30 vertices with  $\Delta = 4$  triangles and  $\Xi = 124$  quadrangles through every vertex. Take  $C(3, -1)$ , which is a positive semidefinite integral matrix of rank two with diagonal 4. Thus  $C$  can only have entries  $-4, -3, \dots, 3$  and  $4$ .

Note that since there are 4 triangles through a vertex, it follows that if  $x$  and  $y$  are adjacent then  $A_{xy}^2 \leq 4$ .

Now suppose that  $C_{xy} = 0$  for some vertices  $x$  and  $y$ . Let  $z$  be another arbitrary vertex. Since  $C$  has rank two it follows that the principal submatrix of  $C$  on vertices  $x, y$  and  $z$  has zero determinant, and so  $C_{xz} = 0$  or  $\pm 4$ . This implies that if  $x$  and  $z$  are adjacent then  $A_{xz}^2 = 0$  or  $4$ , and if  $x$  and  $z$  are not adjacent then  $A_{xz}^2 = 2$  or  $6$ . But then the  $(\sigma, \tau)$ -system has no solution, so  $C$  has no entries  $0$ . Similarly we can show that  $C$  has no entries  $\pm 1$  and  $\pm 3$ . Thus  $C$  only has entries  $\pm 2$  and  $\pm 4$ . This implies that if  $x$  and  $y$  are adjacent then  $A_{xy}^2 = 0$  or  $2$ , and if  $x$  and  $y$  are not adjacent, then  $A_{xy}^2 = 0, 4$  or  $6$ . The  $(\sigma, \tau)$ -system now has one solution  $\sigma_0 = 5, \sigma_2 = 4, \tau_0 = 6, \tau_4 = 10, \tau_6 = 4$ . Now it is not hard to show that a graph with these parameters does not exist.  $\square$

It can be proved in the same way that there is no graph with spectrum  $\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$ . In fact, this case is even easier, since here none of the  $(\sigma, \tau)$ -systems has a solution.

**PROPOSITION 8.** *There is no graph with spectrum  $\{[13]^1, [3 + 2\sqrt{10}]^2, [-1]^{25}, [3 - 2\sqrt{10}]^2\}$ .*

*Proof.* Such a graph is 13-regular on 30 vertices with  $\Delta = 62$  triangles and  $\Xi = 570$  quadrangles through every vertex. Take the matrix  $C(3 + 2\sqrt{10}, -1)$ , so  $C = A^2 - (2 + 2\sqrt{10})A - (3 + 2\sqrt{10})I - \frac{7}{15}(10 - 2\sqrt{10})J$ , which is a positive semidefinite matrix of rank two with diagonal  $\frac{8}{15}(10 - 2\sqrt{10})$ . From this it follows that if  $A_{xy} = 1$  then  $A_{xy}^2 = 9, 10, 11$  or  $12$ , and if  $A_{xy} = 0$  then  $A_{xy}^2 = 0, 1, 2$  or  $3$ . For a non-negative integral solution of the  $(\sigma, \tau)$ -system we have  $\sigma_9 \geq 6$  and  $\sigma_{12} \leq 2$ . Now fix a vertex  $x$ , and let  $y$  and  $z$  be two vertices with  $A_{xy}^2 = A_{xz}^2 = 9$ , then  $C_{xy} = C_{xz} = \frac{8}{15}(10 - 2\sqrt{10}) - 3$ . Since the principal submatrix on the vertices  $x, y$  and  $z$  has zero determinant, it follows that  $C_{yz} = \frac{8}{15}(10 - 2\sqrt{10})$ , so  $A_{yz}^2 = 12$ . For fixed  $y$  we have at least 5 choices for  $z$  left ( $\sigma_9 \geq 6$ ), so for  $y$  we have  $\sigma_{12} \geq 5$ , which is a contradiction.  $\square$

We finish by giving a case where we use the same technique as in the uniqueness proof of the graph  $K_{3,3} \oplus K_3$ .

**PROPOSITION 9.** *There is no graph with spectrum  $\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$ .*

*Proof.* Here we have a 6-regular graph on 21 vertices with  $\Delta = 5$  triangles and  $\Xi = 20$  quadrangles through every vertex. Here we take the matrix  $C(3, -1)$ , then  $C$  is a positive semidefinite integral matrix of rank two with row sums zero and diagonal 2. Thus  $C$  is the Gram matrix of a set of vectors in  $\mathbb{R}^2$  of length  $\sqrt{2}$  such that their inner products are  $\pm 2, \pm 1$  or  $0$ . Note that not both  $0$  and  $\pm 1$  can occur as inner product, since then also inner products that are not allowed occur.

Suppose that inner product  $0$  occurs. Without loss of generality we assume that there is a vertex represented by vector  $\sqrt{2}(1, 0)^T$ . The only vectors that can occur now are  $\pm\sqrt{2}(1, 0)^T$  and  $\pm\sqrt{2}(0, 1)^T$ . Since  $C$  has row sums zero, it follows that the number of vertices

represented by  $\sqrt{2}(1, 0)^T$  equals the number of vertices represented by  $-\sqrt{2}(1, 0)^T$ , and the number of vertices represented by  $\sqrt{2}(0, 1)^T$  equals the number of vertices represented by  $-\sqrt{2}(0, 1)^T$ . But the number of vertices is odd, which is a contradiction.

It follows that if  $x$  and  $y$  are adjacent then  $A_{xy}^2 = 1, 2, 4$  or  $5$  and if  $x$  and  $y$  are not adjacent then  $A_{xy}^2 = 0, 2$  or  $3$ . Now we have the  $(\sigma, \tau)$ -system

$$\begin{array}{rclclcl} \sigma_1 + \sigma_2 + \sigma_4 + \sigma_5 & = & k & = & 6, \\ \sigma_1 + 2\sigma_2 + 4\sigma_4 + 5\sigma_5 & = & 2\Delta & = & 10, \\ \tau_0 + \tau_2 + \tau_3 & = & v - k - 1 & = & 14, \\ 2\tau_2 + 3\tau_3 & = & k(k-1) - 2\Delta & = & 20, \\ \sigma_2 + 6\sigma_4 + 10\sigma_5 + \tau_2 + 3\tau_3 & = & \Xi & = & 20, \end{array}$$

which has three solutions:   
*i.*  $\sigma_5 = 1, \sigma_4 = 0, \sigma_2 = 0, \sigma_1 = 5, \tau_3 = 0, \tau_2 = 10, \tau_0 = 4.$    
*ii.*  $\sigma_5 = 0, \sigma_4 = 1, \sigma_2 = 1, \sigma_1 = 4, \tau_3 = 2, \tau_2 = 7, \tau_0 = 5.$    
*iii.*  $\sigma_5 = 0, \sigma_4 = 0, \sigma_2 = 4, \sigma_1 = 2, \tau_3 = 4, \tau_2 = 4, \tau_0 = 6.$

By looking at our vector representation we see that if there is a vertex for which we are in case *ii*, then there are vertices (those represented by vectors opposite to the vector representing our original vertex) for which the  $(\sigma, \tau)$ -system does not hold. Similarly, if there is a vertex for which we are in case *iii*, then there must be vertices for which we are in case *i*.

Thus we may assume that there is a vertex  $x$  for which we are in case *i*. Let  $y$  be the vertex adjacent to  $x$  with  $A_{xy}^2 = 5$ , then the other neighbours of  $x$  and  $y$  are the same, say  $1, 2, 3, 4$  and  $5$ . Now  $A_{xi}^2 = 1$  for  $i = 1, \dots, 5$ , so  $C_{xi} = -2$ , and  $i$  and  $j$  are not adjacent, for all  $i, j = 1, \dots, 5$ . From the principal submatrix of  $C$  on vertices  $x, i$  and  $j$  it follows that  $C_{ij} = 2$ , but then  $A_{ij}^2 = 3$ , so besides  $x$  and  $y$ ,  $i$  and  $j$  only have one common neighbour. This implies that we can identify the 10 vertices  $z$  not adjacent to  $x$  such that  $A_{xz}^2 = 2$  with the pairs  $\{i, j\}$ ,  $i, j = 1, \dots, 5$ ,  $i \neq j$ , in such a way that  $i$  and  $j$  are adjacent to  $\{i, j\}$ . From the principal submatrix of  $C$  on vertices  $x, i, \{j, k\}$ , with  $i \neq j, k$ , it follows that  $C_{i\{j, k\}} = -1$ , and so  $A_{i\{j, k\}}^2 = 0$ . This implies that the subgraph on the pairs  $\{i, j\}$ ,  $i, j = 1, \dots, 5$  is empty, so that all 10 pairs must be adjacent to the remaining four vertices, which is a contradiction. Thus we may conclude that there is no graph with the given spectrum.  $\square$

## 6. Computer results

When the existence or full classification of a graph with a given set of four eigenvalues could not be determined without the use of a computer, we used basically the same methods as [7]. In some situations, however, it was still not possible to classify the graphs completely on account of the vast amount of CPU time required. Indeed, in some cases we were unable to discover whether or not a graph existed at all.

As before, we used two programs to determine the graphs, one for the case when all the

eigenvalues were integral and another when only two of the eigenvalues were integers. Both methods, however, had a common element, which we now briefly describe. Let  $A$  be the adjacency matrix of a graph  $G$  on  $v$  vertices. In our (backtracking) search for zero-one (symmetric) matrices  $A$  with four eigenvalues we have to ensure, as far as possible, that we avoid a path of the search tree that would yield an isomorphic copy of a graph that had already been discovered. This we did by demanding that the matrix  $A$  be in *standard form*. Thus  $A$  is the greatest adjacency matrix amongst the adjacency matrices of all graphs isomorphic to  $G$ . Here the ordering involved is the lexicographical one on the binary integer obtained by concatenating the rows of the upper triangular part of  $A$ . A simple observation is that if  $r$  rows of  $A$  have been determined in the form

$$\begin{pmatrix} A_r & N_r \\ N_r^T & O \end{pmatrix},$$

where  $A_r$  is a principal sub-matrix of  $A$  of order  $r$ , then this matrix itself must be in standard form. Testing to see that this was the case was very efficient, at least for small values of  $r$ .

To use the eigenvalues of the given graph in our search, observe that, as outlined in Section 3.1, we can use the eigenvalue  $k$  and two of the other three eigenvalues to determine constants  $a, b, c$  and  $d$  such that the matrix  $B$  defined by

$$B = aA^2 + bA + cI + dJ$$

is positive semi-definite and has rank equal to the multiplicity,  $\rho$  say, of the remaining eigenvalue. This can be done for each of the three possible choices of two eigenvalues. The matrix  $B$  then will have two eigenvalues, the non-zero one of which we denote by  $\theta$ . It is then clear that every principal sub-matrix of  $B$  must have rank at most  $\rho$  and have eigenvalues, all of which lie between 0 and  $\theta$ . Thus, for all  $r \leq v$  the matrix  $B_r$  defined by

$$B_r = a(A_r^2 + N_r N_r^T) + bA_r + cI + dJ$$

must satisfy these two conditions.

The method used for testing the rank of  $B_r$  depended on whether the constants  $a, b, c$  and  $d$  above could all be chosen as integers. This is certainly the case when all the eigenvalues of  $G$  are integers, but not always so when only two of the eigenvalues are integral. For details as to how the methods differed the reader is referred to [7]. In the cases when  $\rho$  was small (no more than 5) testing the rank condition was reasonably efficient, and when used in conjunction with the above bounds on the eigenvalues of  $B_r$ , it generally enabled the computer search to be completed. However, other determining factors in the completion of the computer searches were the number of triangles and upper bounds to  $\lambda$  and  $\mu$  (the number of common neighbours of two adjacent and non-adjacent vertices, respectively). The smaller these numbers were, the more likely it was that the

classification was feasible, but this was not always so. Overall, there were eleven sets of four integral eigenvalues and nine sets in which just two of the eigenvalues are integers, on which our computer search made no real impression. Thus the existence of a graph in each of these twenty cases is still in doubt. There were several other instances where a complete search using the ideas described above, was impossible, notably numbers 105, 110, 112, 157, 162, 164 and 170 of Appendix I. An incomplete search in case 162 found 1487 graphs. In fact, these are all graphs having a (Hoffman-)clique of size six, which can be partitioned into three pairs of vertices, such that for each pair there are eight vertices outside the clique that are adjacent to both its vertices and to no other vertices of the clique. The induced subgraph on the vertices outside the clique then is a graph of case 91, and every one of the 28 graphs of this case actually appears in this way. Some lower bounds were obtained in all other cases, except 105, using results of the second author [10, 11]. In [10] a partial classification of all strongly regular graphs with parameters  $(40, 12, 2, 4)$  found 27 such graphs, and each of these was searched for co-cliques of size 10, giving eleven graphs corresponding to number 130. An exhaustive search, that was finally completed, showed that these eleven graphs constituted the complete set. In [11] a partial classification of regular two-graphs on 36 vertices was made and all strongly regular graphs in their switching classes determined. They have parameters  $(36, 15, 6, 6)$  and  $(36, 21, 12, 12)$ , and using them and their complements, together with the complements of the *descendants* of the regular two-graphs (the strongly regular graphs with parameters  $(35, 16, 6, 8)$ ), and searching for cliques or co-cliques of suitable size, enabled the quoted lower bounds to be obtained for the remaining numbers cited above, except for number 105. It seems rather surprising that of the 32,548 graphs found in [11] with parameters  $(36, 14, 4, 6)$  only two possess a co-clique of size 8. These two give rise to the lower bound for number 110.

Overall there were so many graphs found that it is not possible to list them all in this paper. However, in the cases where there were graphs discovered by the computer that were not already known, we list one graph as a representative of these new graphs. Further, all new *transitive* graphs are also listed.

In Appendix I, where there is a table of feasible spectra for regular graphs with four integral eigenvalues, the numbers for which a computer was needed to establish the existence of at least one graph, are 37, 69, 70, 71, 78, 83, 100, 110, 130, 144, 157, 164, and 170. In all of these cases except 70, 100 and 110, there are transitive graphs. Indeed, in some instances there are several such graphs. For example, number 78 has four transitive graphs, number 83 has three, and numbers 71, 157 and 170 have two each. New graphs were also discovered with spectra numbered 39, 44, 48, 49, 54, 72, 104, 138, 148, and 154. In each of these cases, at least one graph was already known. The corresponding graphs are listed at the end of Appendix I. The graphs corresponding to numbers 55, 97, 99, 125 and 129 are already contained in [7] and as a consequence are not included here.

For spectra containing only two integral eigenvalues, there was precisely one number in the listing of Appendix II, namely 23, for which the computer established the existence of a graph when none was already known. In this case there is a unique graph, and it too is transitive. New graphs were also found for spectra numbered 25 and 28. Numbers 45 and



54 were considered in [7], where the quoted lower bounds were obtained. However, none of the graphs coming from number 45 was listed there, and so, for the sake of completeness, we include here those transitive graphs found. The graphs in all these cases are given in Appendix II.

Corresponding to each graph listed there are two lines of data. The first contains the *hexadecimal* form of the graph and the order of its automorphism group. The hexadecimal form is obtained by expressing the binary integer, derived by concatenating the rows of the upper triangular part of the adjacency matrix, as a hexadecimal integer. The next line contains the orbits, unless the graph is transitive.

The reader who wishes to have copies of any, or indeed all, of the graphs found in the above investigation, may obtain them by accessing the second author's home page on the Internet at <http://www.maths.gla.ac.uk/~ted2/>.

## Appendix

In this appendix we list all feasible spectra for connected regular graphs with four eigenvalues and at most 30 vertices. If both the spectrum and its complementary spectrum correspond to connected graphs then only the one with least degree is mentioned. # denotes the number of graphs. In between brackets the number of such graphs or their complements that are a relation in a three-class association scheme is denoted (if positive). The references refer to the sections or the literature.

### I. Four integral eigenvalues

Nr	$\nu$	spectrum	$\Delta$	$\Xi$	#	Notes	References
1	6	$\{[2]^1, [1]^2, [-1]^2, [-2]^1\}$	0	0	1 (1)	$C_6$	4.1
2	8	$\{[5]^1, [1]^2, [-1]^4, [-3]^1\}$	6	22	1 (1)	$\overline{G} = 2C_4$	3.2
3	8	$\{[3]^1, [1]^3, [-1]^3, [-3]^1\}$	0	3	1 (1)	Cube	4.1
4	10	$\{[4]^1, [1]^4, [-1]^4, [-4]^1\}$	0	12	1 (1)	$IG(5, 4, 3)$	4.1
5	12	$\{[9]^1, [1]^3, [-1]^6, [-3]^2\}$	28	204	1 (1)	$\overline{G} = 3C_4$	3.2
6	12	$\{[8]^1, [2]^2, [-1]^8, [-4]^1\}$	19	123	1 (1)	$\overline{G} = 2K_{3,3}$	3.2
7	12	$\{[4]^1, [2]^3, [0]^3, [-2]^5\}$	2	2	2	$L(\text{Cube}), \text{BCS}_9$	3.3
8	12	$\{[7]^1, [1]^4, [-1]^6, [-5]^1\}$	9	81	1 (1)	$\overline{G} = 2CP(3)$	3.2
9	12	$\{[5]^1, [1]^5, [-1]^5, [-5]^1\}$	0	30	1 (1)	$IG(6, 5, 4)$	4.1
10	12	$\{[5]^1, [1]^3, [0]^6, [-4]^2\}$	0	25	0	$\lambda_1 = 1$	3.3
11	12	$\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$	2	14	1	$GQ(2, 2) \setminus 3\text{-clique},$ $\overline{L(CP(3))}$	3.3, 4.3
12	12	$\{[5]^1, [3]^2, [-1]^8, [-3]^1\}$	6	14	1	$C_6 \otimes J_2$	4.2
13	12	$\{[5]^1, [2]^2, [1]^3, [-2]^6\}$	4	9	1 (1)	$L(K_{3,4})$	3.3
14	14	$\{[6]^1, [1]^6, [-1]^6, [-6]^1\}$	0	60	1 (1)	$IG(7, 6, 5)$	4.1
15	15	$\{[4]^1, [2]^5, [-1]^4, [-2]^5\}$	2	0	1 (1)	$L(\text{Petersen})$	3.3, 4.1
16	15	$\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$	4	4	0	$\lambda_3 = -2$	3.3
17	15	$\{[6]^1, [1]^6, [0]^5, [-4]^3\}$	1	36	0	$\lambda_1 = 1$	3.3
18	15	$\{[6]^1, [3]^2, [1]^4, [-2]^8\}$	7	20	1 (1)	$L(K_{3,5})$	3.3
19	16	$\{[13]^1, [1]^4, [-1]^8, [-3]^3\}$	66	738	1 (1)	$\overline{G} = 4C_4$	3.2
20	16	$\{[11]^1, [3]^2, [-1]^{12}, [-5]^1\}$	39	367	1 (1)	$\overline{G} = 2K_{4,4}$	3.2
21	16	$\{[6]^1, [4]^2, [0]^6, [-2]^7\}$	9	27	0	$\lambda_3 = -2$	3.3

22	16	$\{[9]^1, [1]^6, [-1]^8, [-7]^1\}$	12	204	1 (1)	$\overline{G} = 2CP(4)$	3.2
23	16	$\{[7]^1, [1]^7, [-1]^7, [-7]^1\}$	0	105	1 (1)	$IG(8,7,6)$	4.1
24	16	$\{[7]^1, [1]^8, [-1]^5, [-5]^2\}$	3	69	0	$\lambda_1 = 1$	3.3
25	16	$\{[7]^1, [3]^3, [-1]^{11}, [-5]^1\}$	9	57	1	$Cube \otimes J_2$	4.2
26	18	$\{[14]^1, [2]^3, [-1]^{12}, [-4]^2\}$	73	894	1 (1)	$\overline{G} = 3K_{3,3}$	3.2
27	18	$\{[13]^1, [1]^6, [-1]^9, [-5]^2\}$	54	666	1 (1)	$\overline{G} = 3CP(3)$	3.2
28	18	$\{[13]^1, [1]^8, [-2]^8, [-5]^1\}$	56	652	1 (1)	$\overline{G} = 2L_2(3)$	3.2
29	18	$\{[5]^1, [2]^6, [-1]^9, [-4]^2\}$	1	12	1	$K_{3,3} \otimes K_3$	4.2
30	18	$\{[5]^1, [2]^7, [-1]^1, [-2]^9\}$	3	2	0	$\lambda_3 = -2$	3.3
31	18	$\{[11]^1, [2]^4, [-1]^{12}, [-7]^1\}$	28	360	1 (1)	$\overline{G} = 2K_{3,3,3}$	3.2
32	18	$\{[6]^1, [3]^4, [0]^4, [-2]^9\}$	7	16	1	$L(L_2(3))$	3.3
33	18	$\{[7]^1, [1]^{11}, [-2]^4, [-5]^2\}$	2	58	1	$\overline{BCS}_{179}$	3.3
34	18	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$	11	40	2 (1)	$L(K_{3,6}), BCS_{70}$	3.3
35	18	$\{[8]^1, [1]^8, [-1]^8, [-8]^1\}$	0	168	1 (1)	$IG(9,8,7)$	4.1
36	18	$\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$	12	68	0		3.1
37	18	$\{[8]^1, [2]^6, [-1]^8, [-4]^3\}$	10	78	2		6
38	18	$\{[8]^1, [5]^2, [-1]^{14}, [-4]^1\}$	19	96	1	$C_6 \otimes J_3$	4.2
39	18	$\{[8]^1, [2]^4, [0]^9, [-4]^4\}$	8	84	3 (1)	$L_2(3) \otimes J_2$	4.2, 6
40	18	$\{[8]^1, [4]^3, [-1]^8, [-2]^6\}$	18	78	0	$\lambda_3 = -2$	3.3
41	20	$\{[17]^1, [1]^5, [-1]^{10}, [-3]^4\}$	120	1816	1 (1)	$\overline{G} = 5C_4$	3.2
42	20	$\{[16]^1, [1]^8, [-2]^{10}, [-4]^1\}$	99	1401	1 (1)	$\overline{G} = 2Petersen$	3.2
43	20	$\{[14]^1, [4]^2, [-1]^{16}, [-6]^1\}$	66	817	1 (1)	$\overline{G} = 2K_{5,5}$	3.2
44	20	$\{[6]^1, [2]^5, [0]^{10}, [-4]^4\}$	0	27	2 (1)	$Petersen \otimes J_2$	4.2, 6
45	20	$\{[6]^1, [3]^4, [1]^4, [-2]^{11}\}$	6	12	1	$L(IG(5,4,3))$	3.3
46	20	$\{[13]^1, [1]^{10}, [-2]^8, [-7]^1\}$	45	615	1 (1)	$\overline{G} = 2Petersen$	3.2
47	20	$\{[7]^1, [2]^4, [0]^{12}, [-5]^3\}$	0	63	0		6
48	20	$\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$	6	30	9	$SR(26,10,3,4) \setminus 6\text{-cocl.}$	4.1, 4.3, 6
49	20	$\{[7]^1, [3]^5, [-1]^{10}, [-3]^4\}$	9	33	4 (1)	$Dodecahedron_{3,5}$	4.2, 6
50	20	$\{[7]^1, [3]^3, [2]^4, [-2]^{12}\}$	9	27	1 (1)	$Petersen \otimes J_2$	4.2, 6
51	20	$\{[11]^1, [1]^8, [-1]^{10}, [-9]^1\}$	15	415	1 (1)	$L(K_{4,5})$	3.3
52	20	$\{[9]^1, [1]^9, [-1]^9, [-9]^1\}$	0	252	1 (1)	$\overline{G} = 2CP(5)$	3.2
53	20	$\{[9]^1, [3]^4, [-1]^{14}, [-7]^1\}$	12	156	1 (1)	$IG(10,9,8)$	4.1
54	20	$\{[9]^1, [2]^8, [-1]^4, [-3]^7\}$	15	105	1	$IG(5,4,3) \otimes J_2$	4.2
55	20	$\{[9]^1, [3]^5, [-1]^9, [-3]^5\}$	18	108	26	$L_3(5) \setminus 5\text{-coclique}$	4.3, 6
56	21	$\{[6]^1, [3]^5, [-1]^{13}, [-4]^2\}$	5	20	9 (1)	$J(6,3)$	4.1, [7]
57	21	$\{[8]^1, [5]^2, [1]^6, [-2]^{12}\}$	16	72	0		5
58	22	$\{[10]^1, [1]^{10}, [-1]^{10}, [-10]^1\}$	0	360	1 (1)	$L(K_{3,7})$	3.3
59	24	$\{[21]^1, [1]^6, [-1]^{12}, [-3]^5\}$	190	3630	1 (1)	$IG(11,10,9)$	4.1
60	24	$\{[20]^1, [2]^4, [-1]^{16}, [-4]^3\}$	163	2961	1 (1)	$\overline{G} = 6C_4$	3.2
61	24	$\{[19]^1, [3]^3, [-1]^{18}, [-5]^2\}$	139	2395	1 (1)	$\overline{G} = 4K_{3,3}$	3.2
62	24	$\{[19]^1, [1]^8, [-1]^{12}, [-5]^3\}$	135	2403	1 (1)	$\overline{G} = 3K_{4,4}$	3.2
63	24	$\{[5]^1, [3]^6, [-1]^{14}, [-3]^3\}$	4	6	1 (1)	$\overline{G} = 4CP(3)$	3.2
64	24	$\{[5]^1, [2]^8, [0]^8, [-3]^7\}$	0	5	1	$2\text{-cover } C_6 \otimes J_2$	4.4, 6
65	24	$\{[17]^1, [5]^2, [-1]^{20}, [-7]^1\}$	100	1536	0		6
66	24	$\{[6]^1, [4]^4, [0]^8, [-2]^{11}\}$	8	19	1 (1)	$\overline{G} = 2K_{6,6}$	3.2
67	24	$\{[17]^1, [1]^9, [-1]^{12}, [-7]^2\}$	88	1560	0	$\lambda_3 = -2$	3.3
68	24	$\{[7]^1, [3]^6, [-1]^{15}, [-5]^2\}$	5	41	1 (1)	$\overline{G} = 3CP(4)$	3.2
69	24	$\{[7]^1, [3]^3, [1]^{11}, [-3]^9\}$	4	25	0		6
70	24	$\{[8]^1, [2]^{11}, [-2]^9, [-4]^3\}$	7	48	5		6
71	24	$\{[8]^1, [2]^8, [0]^9, [-4]^6\}$	4	60	5		6
72	24	$\{[8]^1, [4]^3, [0]^{15}, [-4]^5\}$	8	68	4		6
73	24	$\{[15]^1, [3]^4, [-1]^{18}, [-9]^1\}$	57	981	5	$L(Cube) \otimes J_2, BCS_9 \otimes J_2$	6
74	24	$\{[8]^1, [4]^3, [2]^5, [-2]^{15}\}$	13	48	1 (1)	$\overline{G} = 2K_{4,4,4}$	3.2
75	24	$\{[9]^1, [1]^{17}, [-3]^2, [-5]^4\}$	4	116	1 (1)	$L(K_{4,6})$	3.3
76	24	$\{[9]^1, [3]^7, [-1]^9, [-3]^7\}$	15	84	1	$GQ(2,4) \setminus 3\text{-cl}, \overline{BCS}_{183}$	3.3, 4.3
77	24	$\{[14]^1, [2]^6, [-1]^{16}, [-10]^1\}$	37	822	?		
78	24	$\{[9]^1, [3]^4, [1]^9, [-3]^{10}\}$	12	84	1 (1)	$\overline{G} = 2K_{3,3,3,3}$	3.2
79	24	$\{[9]^1, [6]^2, [1]^7, [-2]^{14}\}$	22	119	87		6
80	24	$\{[10]^1, [2]^3, [0]^{18}, [-8]^2\}$	0	285	1 (1)	$L(K_{3,8})$	3.3
81	24	$\{[10]^1, [1]^{16}, [-2]^4, [-6]^3\}$	7	196	0	$\lambda_1 = 1$	3.3
82	24	$\{[10]^1, [4]^2, [0]^{18}, [-6]^3\}$	10	205	0		5
83	24	$\{[10]^1, [2]^{11}, [-2]^8, [-4]^4\}$	16	141	0		6
84	24	$\{[10]^1, [4]^5, [0]^3, [-2]^{15}\}$	25	145	183	$\lambda_3 = -2$	3.3
85	24	$\{[13]^1, [1]^{10}, [-1]^{12}, [-11]^1\}$	18	738	1 (1)	$\overline{G} = 2CP(6)$	3.2
86	24	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$	24	141	9	$L(CP(4)), BCS_{153-160}$	3.3
87	24	$\{[11]^1, [1]^{11}, [-1]^{11}, [-11]^1\}$	0	495	1 (1)	$IG(12,11,10)$	4.1
88	24	$\{[11]^1, [3]^5, [-1]^{17}, [-9]^1\}$	15	335	1	$IG(6,5,4) \otimes J_2$	4.2
89	24	$\{[11]^1, [5]^3, [-1]^{19}, [-7]^1\}$	28	279	1	$Cube \otimes J_3$	4.2
90	24	$\{[11]^1, [1]^{16}, [-1]^2, [-5]^5\}$	15	255	0	$\lambda_1 = 1$	3.3
91	24	$\{[11]^1, [3]^6, [-1]^{14}, [-5]^3\}$	23	239	28	$(GQ(2,2) \setminus 3\text{-clique}) \otimes J_2$	4.2, 6

92	24	$\{[111]^1, [7]^2, [-1]^{20}, [-5]^1\}$	39	303	1	$C_6 \otimes J_4$	4.2
93	24	$\{[111]^1, [3]^7, [-1]^8, [-3]^8\}$	27	215	?		
94	25	$\{[10]^1, [5]^2, [0]^{18}, [-5]^4\}$	15	180	0		5
95	26	$\{[12]^1, [1]^{12}, [-1]^{12}, [-12]^1\}$	0	660	1 (1)	$IG(13,12,11)$	4.1
96	27	$\{[22]^1, [1]^{12}, [-2]^{12}, [-5]^2\}$	191	3892	1 (1)	$\overline{G} = 3L_2(3)$	3.2
97	27	$\{[6]^1, [3]^6, [0]^{12}, [-3]^8\}$	3	12	4 (1)	$H(3,3), 3\text{-cover } C_3 \otimes J_3$	4.1, 4.4, [7]
98	27	$\{[20]^1, [2]^6, [-1]^{18}, [-7]^2\}$	136	2664	1 (1)	$\overline{G} = 3K_{3,3,3}$	3.2
99	27	$\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$	4	48	13 (3)	$GQ(2,4) \setminus \text{spread } (2 \times)$ $H(3,3)_3, GQ(3,3)_2(x)$	4.1, 4.3, [7]
100	27	$\{[8]^1, [5]^4, [-1]^{20}, [-4]^2\}$	16	72	1		6
101	27	$\{[10]^1, [4]^6, [1]^2, [-2]^{18}\}$	23	124	1	$L(K_{3,3,3})$	3.3
102	27	$\{[10]^1, [7]^2, [1]^8, [-2]^{16}\}$	29	184	1 (1)	$L(K_{3,9})$	3.3
103	27	$\{[12]^1, [3]^2, [0]^{22}, [-9]^2\}$	6	492	0		5
104	27	$\{[12]^1, [3]^4, [0]^{18}, [-6]^4\}$	18	348	5 (1)	$L_2(3) \otimes J_3$	4.2, 6
105	27	$\{[12]^1, [3]^8, [0]^6, [-3]^{12}\}$	30	276	$\geq 1$ (1)	$H(3,3)_2$	4.1
106	28	$\{[25]^1, [1]^7, [-1]^{14}, [-3]^6\}$	276	6372	1 (1)	$\overline{G} = 7C_4$	3.2
107	28	$\{[6]^1, [5]^4, [-1]^{20}, [-2]^3\}$	12	36	0	$\lambda_3 = -2$	3.3
108	28	$\{[20]^1, [6]^2, [-1]^{24}, [-8]^1\}$	141	2587	1 (1)	$\overline{G} = 2K_{7,7}$	3.2
109	28	$\{[9]^1, [5]^3, [2]^6, [-2]^{18}\}$	18	81	1 (1)	$L(K_{4,7})$	3.3
110	28	$\{[10]^1, [2]^{14}, [-2]^7, [-4]^6\}$	12	117	$\geq 2$	$SR(36,14,4,6) \setminus 8\text{-coc1.}$	4.3, 6
111	28	$\{[11]^1, [3]^7, [1]^7, [-3]^{13}\}$	21	175	?		
112	28	$\{[12]^1, [2]^{14}, [-2]^6, [-4]^7\}$	24	270	$\geq 8472$ (56)	$\overline{T(8)} \setminus \text{spread}$ $SR(35,16,6,8) \setminus 7\text{-coc1.}$	4.3, 6
113	28	$\{[15]^1, [1]^{12}, [-1]^{14}, [-13]^1\}$	21	1197	1 (1)	$\overline{G} = 2CP(7)$	3.2
114	28	$\{[13]^1, [1]^{13}, [-1]^{13}, [-13]^1\}$	0	858	1 (1)	$IG(14,13,12)$	4.1
115	28	$\{[13]^1, [3]^6, [-1]^{20}, [-11]^1\}$	18	618	1	$IG(7,6,5) \otimes J_2$	4.2
116	28	$\{[13]^1, [5]^5, [-1]^6, [-2]^{16}\}$	48	408	0	$\lambda_3 = -2$	3.3
117	30	$\{[26]^1, [2]^5, [-1]^{20}, [-4]^4\}$	289	6972	1 (1)	$\overline{G} = 5K_{3,3}$	3.2
118	30	$\{[26]^1, [1]^{12}, [-2]^{15}, [-4]^2\}$	289	6966	1 (1)	$\overline{G} = 3\text{Petersen}$	3.2
119	30	$\{[25]^1, [1]^{10}, [-1]^{15}, [-5]^4\}$	252	5940	1 (1)	$\overline{G} = 5CP(3)$	3.2
120	30	$\{[24]^1, [4]^3, [-1]^{24}, [-6]^2\}$	226	5022	1 (1)	$\overline{G} = 3K_{5,5}$	3.2
121	30	$\{[6]^1, [2]^9, [1]^9, [-3]^{11}\}$	0	6	0		5
122	30	$\{[23]^1, [2]^{10}, [-2]^{18}, [-7]^1\}$	196	4194	1 (1)	$\overline{G} = 2GQ(2,2)$	3.2
123	30	$\{[6]^1, [3]^8, [1]^4, [-2]^{17}\}$	5	4	0	$\lambda_3 = -2$	3.3
124	30	$\{[23]^1, [1]^{15}, [-2]^{12}, [-7]^2\}$	190	4230	1 (1)	$\overline{G} = 3\text{Petersen}$	3.2
125	30	$\{[7]^1, [2]^{14}, [-2]^{14}, [-7]^1\}$	0	42	4 (4)	$IG(15,7,3)$	4.1, [7]
126	30	$\{[7]^1, [2]^{15}, [-2]^5, [-3]^9\}$	3	12	0		5
127	30	$\{[7]^1, [4]^5, [0]^{15}, [-3]^9\}$	7	28	0		6
128	30	$\{[7]^1, [2]^{12}, [1]^5, [-3]^{12}\}$	2	14	0		5
129	30	$\{[8]^1, [2]^{14}, [-2]^{14}, [-8]^1\}$	0	84	4 (4)	$IG(15,8,4)$	4.1, [7]
130	30	$\{[8]^1, [2]^{15}, [-2]^9, [-4]^5\}$	4	36	11	$GQ(3,3) \setminus 10\text{-coclique}$	4.3, 6
131	30	$\{[8]^1, [3]^9, [-1]^{15}, [-4]^5\}$	7	42	0		6
132	30	$\{[8]^1, [4]^7, [-1]^8, [-2]^{14}\}$	14	42	0	$\lambda_3 = -2$	3.3
133	30	$\{[21]^1, [1]^{12}, [-1]^{15}, [-9]^2\}$	130	3030	1 (1)	$\overline{G} = 3CP(5)$	3.2
134	30	$\{[8]^1, [4]^5, [2]^5, [-2]^{19}\}$	12	36	1	$L(IG(6,5,4))$	3.3
135	30	$\{[21]^1, [1]^{18}, [-3]^{10}, [-9]^1\}$	138	2934	1 (1)	$\overline{G} = 2GQ(2,2)$	3.2
136	30	$\{[9]^1, [3]^8, [-1]^{19}, [-7]^2\}$	4	124	0		5
137	30	$\{[9]^1, [4]^6, [-1]^{21}, [-6]^2\}$	11	102	0		5
138	30	$\{[9]^1, [3]^5, [0]^{20}, [-6]^4\}$	0	126	2 (1)	$\text{Petersen} \otimes J_3$	4.2, 6
139	30	$\{[9]^1, [4]^4, [0]^{20}, [-5]^5\}$	6	102	0		6
140	30	$\{[9]^1, [3]^{10}, [-1]^9, [-3]^{10}\}$	12	60	?		
141	30	$\{[9]^1, [5]^5, [-1]^{19}, [-3]^5\}$	20	92	1	$L(\text{Petersen}) \otimes J_2$	4.2, 6
142	30	$\{[9]^1, [7]^3, [-1]^{24}, [-3]^2\}$	28	156	0		5
143	30	$\{[9]^1, [4]^4, [3]^5, [-2]^{20}\}$	16	62	1 (1)	$L(K_{5,6})$	3.3
144	30	$\{[10]^1, [2]^{15}, [-2]^{10}, [-5]^4\}$	9	120	3		6
145	30	$\{[19]^1, [4]^4, [-1]^{24}, [-11]^1\}$	96	2082	1 (1)	$\overline{G} = 2K_{5,5,5}$	3.2
146	30	$\{[10]^1, [5]^4, [2]^5, [-2]^{20}\}$	23	120	1	$L(\text{Petersen})$	3.3
147	30	$\{[11]^1, [2]^{16}, [-3]^9, [-4]^4\}$	16	162	0		6
148	30	$\{[11]^1, [5]^5, [-1]^{20}, [-4]^4\}$	28	198	8 (1)	$\text{Petersen} \otimes J_3$	4.2, 6
149	30	$\{[11]^1, [2]^{10}, [1]^9, [-4]^{10}\}$	13	174	?		
150	30	$\{[11]^1, [6]^4, [-1]^{20}, [-3]^5\}$	34	222	0		6
151	30	$\{[11]^1, [5]^5, [1]^4, [-2]^{20}\}$	30	186	0	$\lambda_3 = -2$	3.3
152	30	$\{[11]^1, [8]^2, [1]^9, [-2]^{18}\}$	37	270	1 (1)	$L(K_{3,10})$	3.3
153	30	$\{[12]^1, [2]^6, [0]^{20}, [-8]^3\}$	4	414	0		6
154	30	$\{[12]^1, [2]^9, [0]^{15}, [-6]^5\}$	12	318	2 (1)	$GQ(2,2) \otimes J_2$	4.2, 6
155	30	$\{[12]^1, [2]^{16}, [-3]^8, [-4]^5\}$	22	244	?		
156	30	$\{[12]^1, [2]^{14}, [0]^5, [-4]^{10}\}$	20	254	?		
157	30	$\{[12]^1, [3]^{10}, [0]^5, [-3]^{14}\}$	27	240	$\geq 66986$	$L_3(6) \setminus 6\text{-coclique}$	4.3, 6
158	30	$\{[17]^1, [2]^8, [-1]^{20}, [-13]^1\}$	46	1590	1 (1)	$\overline{G} = 2K_{3,3,3,3,3}$	3.2
159	30	$\{[12]^1, [4]^5, [1]^{10}, [-3]^{14}\}$	28	248	?		
160	30	$\{[13]^1, [1]^{20}, [-1]^5, [-7]^4\}$	14	474	0	$\lambda_1 = 1$	3.3

161	30	$\{[13]^1, [2]^{15}, [-2]^9, [-5]^5\}$	27	372	?			
162	30	$\{[13]^1, [3]^9, [-1]^{15}, [-5]^5\}$	30	378	$\geq 1487$	(1)	$GQ(2,2) \otimes J_2$	4.2, 6
163	30	$\{[13]^1, [3]^{11}, [-2]^8, [-3]^{10}\}$	36	344	?			
164	30	$\{[13]^1, [3]^9, [1]^5, [-3]^{15}\}$	34	346	$\geq 82$		$L_3(6) \setminus 6\text{-clique}$	4.3, 6
165	30	$\{[14]^1, [1]^{14}, [-1]^{14}, [-14]^1\}$	0	1092	1	(1)	$IG(15,14,13)$	4.1
166	30	$\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$	10	930	0			3.1
167	30	$\{[14]^1, [5]^4, [-1]^{24}, [-10]^1\}$	37	660	1		$IG(5,4,3) \otimes J_3$	4.2
168	30	$\{[14]^1, [4]^6, [-1]^{20}, [-6]^3\}$	41	542	?			
169	30	$\{[14]^1, [9]^2, [-1]^{26}, [-6]^1\}$	66	692	1		$C_6 \otimes J_5$	4.2
170	30	$\{[14]^1, [2]^{15}, [-1]^4, [-4]^{10}\}$	37	498	$\geq 24931$		$SR(35,16,6,8) \setminus 5\text{-cl.}$	4.3, 6

Nr	v	Hexadecimal form / Orbits	Aut
37	18	FF0078E00FC024F09784F84CCB339878F3D4A90	432
39	18	FF0078E0093C24F096696670DCCE7818C3DFE00 (1 2 7 8 9 10 11 12)(3 4 5 6 15 16 17 18)(13 14)	512
44	20	FC0000F8007C0043C0878111C88E00C0C049A4DB00063C28 (1 8 9 12 13 16 19 20)(2 3 4 5 6 7 10 11 14 15 17 18)	3072
48	20	FE001878022381126110B228C1A528B49A9984D154D52C80 (1 5 6 9 10 11 12 13 15 17)(2 3 4 7 8 14 16 18 19 20)	20
49	20	FE001F80043C003C0207801E100F00F998664CCCD52AA548 (1 2 3 4)(5 6 7 8 9 10 11 12)(13 14 15 16 17 18 19 20)	1024
54	20	FF801E1E04C0F00CCD7513AA299C66C0B653B5BA469CB748	40
69	24	FE00018780022380220700401C0924C2495124B000E48C92A89D070AA58C2124604A0	1152
70	24	FF0001E3800003F00030F001998019FCC03A30C21E1E01CC303C1542A892610A52592 (1 2 7 8 9 10 11 12)(3 4 5 6 13 14 15 16 17 18 19 20 21 22 23 24)	384
71	24	FF000103F000007E21C61006394968824AAA91A2D8827088D2544160398922245044F FF000183E00400F800007E318C0318C72C1583C0F2C043A4B0CC121CDA04493294C10	1152 144
72	24	FF0001E3800093C0093C009660259807030700CE07801800300F1E078033331E78CDE (1 2 7 8 9 10 11 12)(3 4 5 6 19 20 21 22)(13 14 15 16 17 18 23 24)	4096
78	24	FF8001E1E004CCC0033C0201F80061F0CCC0C33E18781E42A8156528B599662D4A58C FF8001E1E004CCC0033C0201F80061F0CCC0C33E15582AC30C0F2528B599662D4A58C FF8001C1F0018C7014A0E18C039424C911988E10E4236F12552AAA91256D8C199D119 FF8001E1E00488F0044CC24456088B8949A68662CC34C8A1A5255492C553A3292223F	1152 1152 1152 48
83	24	FFC001E0F800C63C0A5238A52698C6632692C2759B194B04F045CCD2CBE83CCCA8859 FFC001E0F804843E04231E425B084DB36232E1CF913D29079248C8E71352E4A44F28F FFC001E0F804C03E00631D635216AA46919B26A091E8B3A85CC37518F23CBA3DA8908	192 384 48
91	24	FFE001F878043C3C003FC2666606799CC3C333C5554AD5296956952B4A5555861CCFF	384
100	27	FF00003F80001F800010F00003C00081E0001E00403C000F0380718071001C007E1C6 1C40701FF1F11C7C40E (1 2 3 22 23 24 25 26 27)(4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21)	3981312
104	27	FFF0003F07C000C63F018C7E0631F8286E3A86E3D0DC70F8007C0E7C70F8FFE3F0038 038380E07E7EFFF0000 (1 10 11 18)(2 9 12 13 14 15 16 17)(3 4 5 6 7 8 22 23 24 25 26 27)(19 20 21)	995328
110	28	FFC0001E0F8000843F00423CC04245708495A268592645670527C0B184C305168C2AA 4C719330C21F07A868AA498800 (1 2 11 16)(3 4 5 6 7 8 9 10 12 13 14 15 19 20 25 26)(17 18 21 22 23 24 27 28)	32
112	28	FFF0001E03F804C183E030631F10A5B1284DB20B4E4B623A4B54670E7B226E94A8F24 52A923C71239C4D4B92913CA3C	1344
130	30	FF0000040FC0000007E00861070008607124124804924AA030A04A2A050229108382B 40085284692068528105C12982306882232210C0	1440
138	30	FF80000007F80000FF00003FC000181F80181F80303F00A083E5041F5041F00000000 E00E00118E231C8C739C39C00001C0E0EE380890 (1 11 17 18 23 24 28 29)(2 3 4 5 6 7 8 9 10 14 15 16 20 21 22 25 26 27)(12 13 19 30)	17915904
144	30	FFC0000703F0001801F8018061E00C0C8E3300E801872C415270A14DA160BAC0AB306 6031E4B60C6C26424C38842C0AAA0E2E450C0040	60

148	30	FFE00007FE00007FC0000C0FC00207E00007E001803F0200FC0007E06001FC003F000 FF1C711C7038E31C78E38E3D26499B24D6593498 (1 2 3 4 5 6)(7 8 9 10 11 12 13 14 15 16 17 18)(19 20 23 24 26 27 28 29)(21 22 25 30) <b>5971968</b>
154	30	FFF00007E0F8000210FF00843FC0211E1E211E1E2103FC840FF0F8078F8781FF007F3 33E666001E03C0F006661998330CD99980C36600 (1 2 9 10 11 12 13 14 15 16 17 18)(3 4 5 6 7 8 19 20 21 22 23 24 25 26 27 28 29 30) <b>73728</b>
157	30	FFF00007C0FC00661878006199C00C331DAAA092AAA1275052554152C264E0331C8CD 8899C5AA26AA4E4AA1955AA4D49188D70E3188FE FFF00007C0FC0064107E01C390605A262997011C4A69C0C0CCAB2B03124DC461952A9 0C229F295A8967D8141BE228E28AEE5B3901B638 <b>720</b> <b>60</b>
162	30	FFF80007FF8000700FF00C03FC020001FE0001FFC3C3C30F0F08078780787FF00FFC0 3E1FE01FE13330CCC999999C330CF9998A9552A4 (1 2 3 4)(5 6)(7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22)(23 24 25 26 27 28 29 30) <b>65536</b>
164	30	FFF80007E07E00730C3C0C30C0F206066600199EEBA0AADB6812E0D55CDA223299D91 D08AE2362C59AADD0552E8E14D359A74A1D97D38 <b>240</b>
170	30	FFFC0007E03F8073040FC230E38C544D3603AAAF0C2CF93B0CDA8D96AD6C97DC589CD 188B6A8CC7EAE23A430789DFD45A8E3138CD53C8 FFFC0007E03F8063860F881C631C010E7FC909ED494AB49295BE4D87526AA936A3F8C C66AD49D39F3272554D1D3519CF2724C9912E51E <b>60</b> <b>720</b>

## II. Two integral eigenvalues

Nr	$\nu$	spectrum	$\Delta$	$\Xi$	#	Notes	References
1	10	$\{[7]^1, [-3]^1, [0.618]^4, [-1.618]^4\}$	15	80	1 (1)	$\overline{G} = 2C_5$	3.2
2	10	$\{[4]^1, [0]^5, [1.236]^2, [-3.236]^2\}$	0	10	1 (1)	$C_5 \otimes J_2$	4.2
3	12	$\{[5]^1, [-1]^5, [2.236]^3, [-2.236]^3\}$	5	10	1 (1)	Icosahedron	4.1, [7]
4	12	$\{[5]^1, [1]^3, [0.732]^4, [-2.732]^4\}$	2	13	0	$\lambda_1 = 1$	3.3
5	14	$\{[3]^1, [-3]^1, [1.414]^6, [-1.414]^6\}$	0	0	1 (1)	$IG(7,3,1)$	4.1
6	14	$\{[4]^1, [-4]^1, [1.414]^6, [-1.414]^6\}$	0	6	1 (1)	$IG(7,4,2)$	4.1
7	14	$\{[6]^1, [0]^7, [1.646]^3, [-3.646]^3\}$	3	33	0		6
8	15	$\{[12]^1, [-3]^2, [0.618]^6, [-1.618]^6\}$	55	560	1 (1)	$\overline{G} = 3C_5$	3.2
9	15	$\{[4]^1, [0]^6, [1.791]^4, [-2.791]^4\}$	0	4	0		5
10	15	$\{[6]^1, [-1]^{10}, [4.162]^2, [-2.162]^2\}$	11	32	0		5
11	15	$\{[6]^1, [-1]^6, [2.449]^4, [-2.449]^4\}$	7	20	0		6
12	15	$\{[6]^1, [0]^{10}, [1.854]^2, [-4.854]^2\}$	0	48	1 (1)	$C_5 \otimes J_3$	4.2
13	16	$\{[7]^1, [-1]^{11}, [4.464]^2, [-2.464]^2\}$	15	57	0		5
14	18	$\{[5]^1, [3]^1, [1.303]^8, [-2.303]^8\}$	2	4	1	$\text{td}L_2(3)$	4.2
15	20	$\{[17]^1, [-3]^3, [0.618]^8, [-1.618]^8\}$	120	1815	1 (1)	$\overline{G} = 4C_5$	3.2
16	20	$\{[5]^1, [-1]^5, [2.236]^7, [-2.236]^7\}$	3	2	1	2-cover $C_5 \otimes J_2$	4.4, 6
17	20	$\{[7]^1, [-1]^{15}, [5.873]^2, [-1.873]^2\}$	18	75	0	$\lambda_3 > -2$	3.3
18	20	$\{[8]^1, [-2]^9, [3.236]^5, [-1.236]^5\}$	15	60	0	$\lambda_3 = -2$	3.3
19	20	$\{[8]^1, [0]^{15}, [2.472]^2, [-6.472]^2\}$	0	132	1 (1)	$C_5 \otimes J_4$	4.2
20	20	$\{[8]^1, [0]^{11}, [2.317]^4, [-4.317]^4\}$	6	80	0		6
21	21	$\{[4]^1, [-2]^8, [2.414]^6, [-0.414]^6\}$	2	0	1 (1)	$L(IG(7,3,1))$	3.3, 4.1
22	21	$\{[6]^1, [-1]^6, [2.449]^7, [-2.449]^7\}$	5	10	0		6
23	21	$\{[6]^1, [0]^8, [2.193]^6, [-3.193]^6\}$	2	16	1		6
24	21	$\{[8]^1, [-1]^{14}, [4.742]^3, [-2.742]^3\}$	18	78	0		6
25	21	$\{[8]^1, [-1]^8, [2.828]^6, [-2.828]^6\}$	12	56	6 (1)	$L(IG(7,3,1))_3$	4.1, 6
26	21	$\{[8]^1, [1]^{12}, [-0.209]^4, [-4.791]^4\}$	2	88	0	$\lambda_1 = 1$	3.3
27	21	$\{[8]^1, [1]^6, [1.449]^7, [-3.449]^7\}$	6	62	0		5
28	21	$\{[8]^1, [2]^8, [-0.586]^6, [-3.414]^6\}$	8	60	28 (1)	$L(IG(7,3,1))_2$	4.1, 6
29	22	$\{[5]^1, [-5]^1, [1.732]^{10}, [-1.732]^{10}\}$	0	10	1 (1)	$IG(11,5,2)$	4.1
30	22	$\{[5]^1, [0]^{11}, [2.372]^5, [-3.372]^5\}$	0	10	0		6
31	22	$\{[6]^1, [-6]^1, [1.732]^{10}, [-1.732]^{10}\}$	0	30	1 (1)	$IG(11,6,3)$	4.1
32	22	$\{[10]^1, [0]^{11}, [2.317]^5, [-4.317]^5\}$	15	175	0		6

33	24	$\{[7]^1, [-1]^{15}, [4.464]^4, [-2.464]^4\}$	13	41	0		6
34	24	$\{[7]^1, [-1]^7, [2.646]^8, [-2.646]^8\}$	7	21	10 (1)	Klein	4.1, [7]
35	24	$\{[8]^1, [0]^{15}, [2.873]^4, [-4.873]^4\}$	3	78	0		6
36	24	$\{[9]^1, [1]^{15}, [-0.551]^4, [-5.449]^4\}$	2	134	0	$\lambda_1 = 1$	3.3
37	24	$\{[9]^1, [1]^7, [1.646]^8, [-3.646]^8\}$	8	91	1 (1)	Klein <sub>1,3</sub>	4.1, 6
38	24	$\{[11]^1, [-1]^{17}, [5.472]^3, [-3.472]^3\}$	35	255	1	Icosahedron $\otimes J_2$	4.2, 6
39	25	$\{[22]^1, [-3]^4, [0.618]^{10}, [-1.618]^{10}\}$	210	4220	1 (1)	$\overline{G} = 5C_5$	3.2
40	25	$\{[10]^1, [0]^{20}, [3.090]^2, [-8.090]^2\}$	0	280	1 (1)	$C_5 \otimes J_5$	4.2
41	26	$\{[4]^1, [-4]^1, [1.732]^{12}, [-1.732]^{12}\}$	0	0	1 (1)	$IG(13, 4, 1)$	4.1
42	26	$\{[19]^1, [-7]^1, [1.303]^{12}, [-2.303]^{12}\}$	123	2208	1 (1)	$\overline{G} = 2P(13)$	3.2
43	26	$\{[7]^1, [5]^1, [1.562]^{12}, [-2.562]^{12}\}$	6	24	1	$\text{td}P(13)$	4.2
44	26	$\{[9]^1, [-9]^1, [1.732]^{12}, [-1.732]^{12}\}$	0	180	1 (1)	$IG(13, 9, 6)$	4.1
45	26	$\{[12]^1, [0]^{13}, [2.606]^6, [-4.606]^6\}$	24	318	$\geq 85$ (1)	$P(13) \otimes J_2$	4.2, 6, [7]
46	27	$\{[8]^1, [-1]^{20}, [6.243]^3, [-2.243]^3\}$	22	102	0		5
47	27	$\{[8]^1, [-1]^{14}, [3.854]^6, [-2.854]^6\}$	13	48	1	3-cover $C_3 \otimes J_3$	4.4, 6
48	28	$\{[6]^1, [-2]^{15}, [3.414]^6, [0.586]^6\}$	6	9	1	$L(IG(7, 4, 2)),$ Coxeter <sub>4</sub>	3.3, 4.1
49	28	$\{[7]^1, [-1]^7, [2.646]^{10}, [-2.646]^{10}\}$	6	15	0		6
50	28	$\{[9]^1, [-1]^{21}, [6.583]^3, [-2.583]^3\}$	27	144	0		6
51	28	$\{[9]^1, [0]^{21}, [3.623]^3, [-6.623]^3\}$	0	153	0		6
52	28	$\{[12]^1, [0]^{21}, [3.292]^3, [-7.292]^3\}$	12	390	0		6
53	28	$\{[12]^1, [0]^{15}, [2.873]^6, [-4.873]^6\}$	21	300	?		
54	28	$\{[13]^1, [-1]^{13}, [3.606]^7, [-3.606]^7\}$	39	390	$\geq 515$ (1)	Taylor	4.1, [7]
55	30	$\{[27]^1, [-3]^5, [0.618]^{12}, [-1.618]^{12}\}$	325	8150	1 (1)	$\overline{G} = 6C_5$	3.2
56	30	$\{[7]^1, [-3]^9, [2.732]^{10}, [-0.732]^{10}\}$	5	16	?		
57	30	$\{[8]^1, [0]^{21}, [3.583]^4, [-5.583]^4\}$	0	84	0		5
58	30	$\{[9]^1, [3]^5, [1.236]^{12}, [-3.236]^{12}\}$	8	62	?		
59	30	$\{[10]^1, [0]^{19}, [3.359]^5, [-5.359]^5\}$	7	151	0		6
60	30	$\{[11]^1, [-1]^{11}, [3.317]^9, [-3.317]^9\}$	22	165	?		
61	30	$\{[11]^1, [1]^{19}, [0.162]^5, [-6.162]^5\}$	3	249	0	$\lambda_1 = 1$	3.3
62	30	$\{[11]^1, [5]^5, [-0.382]^{12}, [-2.618]^{12}\}$	29	190	?		
63	30	$\{[12]^1, [0]^{25}, [3.708]^2, [-9.708]^2\}$	0	510	1 (1)	$C_5 \otimes J_6$	4.2
64	30	$\{[13]^1, [-2]^{19}, [5.372]^5, [-0.372]^5\}$	47	388	0	$\lambda_3 = -2$	3.3
65	30	$\{[13]^1, [-1]^{25}, [9.325]^2, [-3.325]^2\}$	62	570	0		5
66	30	$\{[13]^1, [-1]^{21}, [5.899]^4, [-3.899]^4\}$	46	410	?		
67	30	$\{[13]^1, [3]^9, [-0.268]^{10}, [-3.732]^{10}\}$	32	358	?		
68	30	$\{[14]^1, [-2]^{21}, [5.791]^4, [1.209]^4\}$	56	532	0	$\lambda_3 = -2$	3.3
69	30	$\{[14]^1, [0]^{15}, [2.873]^7, [-4.873]^7\}$	35	525	?		
70	30	$\{[14]^1, [2]^{11}, [0.449]^9, [-4.449]^9\}$	34	513	?		

Nr	$\nu$	Hexadecimal form / Orbits	Aut
23	21	FC0008780001E04C880066180B130A8051224912B06104B4590A4	42
25	21	FF000F1C01C0E0120E0849824AE4332233C1CB52390BAA691ED18 FF000E1E0066602A8C15430C51489303902E2A542CD47B3B65950	21 336
28	21	FF000E1E0060780A23819A4A27154B358934382E30E88CD2263A8 FF000C1F00843C211991915498B62570C0F4446A4A324B524C93C	42 336
45	26	FFF0007E0F806390F049A23909C1E281FB867A014DF8B98CD630FABF64B6EA25304E6 C5D2BC52ED908 FFF000780FE00E1C3C3870F1326673266607F00730F3987B87E70FC730F9860003CCF 367E78C31FE00	78 638976

### III. One integral eigenvalue

Nr	$\nu$	spectrum	$\Delta$	$\Xi$	#	Notes	References
1	7	$\{[2]^1, [1.247]^2, [-0.445]^2, [-1.802]^2\}$	0	0	1 (1)	$C_7$	3.3, 4.1.1
2	13	$\{[4]^1, [1.377]^4, [0.274]^4, [-2.651]^4\}$	0	4	1 (1)	Cycl(13)	4.1.1
3	19	$\{[6]^1, [2.507]^6, [-1.222]^6, [-2.285]^6\}$	6	12	1 (1)	Cycl(19)	4.1.1
4	28	$\{[9]^1, [2.604]^9, [-0.110]^9, [-3.494]^9\}$	9	72	$\geq 2$ (2)	Mathon, Hollmann	4.1.1

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